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MHD Equilibrium Equation with Azimuthal Rotation in a Curvilinear Coordinate System

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We derive, according to a procedure introduced by Maschke and Perrin, an equation for MHD stationary equilibrium with azimuthal rotation in an orthogonal curvilinear coordinate system. We assume that there is an ignorable coordinate so that surface quantities like the magnetic flux and the rotation frequency do not depend on it. The temperature is also considered a surface quantity. As an application of the formalism, we consider prolate spheroidal coordinates, which are convenient for studying plasma rotation in compact tori configurations like Spheromaks.

1. INTRODUCTION

Experimental results indicate the existence of azimuthal (toroidal) plasma rotation in Tokamaks subjected to neutral beam heating (Bell, 1979; Suckewer *et al.*, 1979). In field-reversed configurations (FRC), azimuthal rotation is responsible for a type of instability that may destroy plasma confinement (Linford *et al.*, 1979). Stellar plasmas are also systems in which azimuthal rotation plays a major role (Plumpton and Ferraro, 1955). One possible approach to investigate the effects of rotation on MHD equilibrium and stability properties would be to obtain numerical solutions of the 3-dimensional ideal MHD equations, but even so we would like to have some analytical solutions in order to benchmark the computer results.

Theoretical investigations of such stationary MHD equilibria (in which all time derivatives vanish, but with a constant angular velocity) are possible since the introduction of a pressure equilibrium equation by Maschke and

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Perrin (1980). Magnetic flux surfaces rotate rigidly with the plasma, according to Alfvén's theorem (Alfvén, 1953), and are characterized by poloidal flux and current functions that satisfy an elliptic partial differential equation which, in the limit of vanishing rotation, is reduced to the Grad–Shafranov equation (Stacey, 1981)

In addition to the ideal MHD equations, Maschke and Perrin have made some thermodynamic assumptions: the plasma is taken to be an ideal gas and its internal energy is proportional to the temperature. Furthermore, they assume that the plasma thermal conductivity is larger along the magnetic field lines than across them, so it would be possible to consider the plasma temperature as a surface quantity. A similar equation has also been derived by considering the entropy as a surface quantity.

However, the equation originally derived by Maschke and Perrin holds only for cylindrical coordinates, and very few analytical and numerical solutions of it are known. Recently Viana *et al.* (1997) derived and analytically solved a form of Maschke–Perrin equation in spherical coordinates, which is a simple model to study FRCs and Spheromak-like configurations.

In this work we derive a version of Maschke–Perrin equation that holds for any orthogonal curvilinear coordinate system in which there is an ignorable coordinate, so that the magnetic flux surfaces are axisymmetric. We have made essentially the same assumptions as Maschke and Perrin, particularly with respect to the role of the temperature. As an example, we consider prolate spheroidal coordinates, which is a convenient coordinate system for some Spheromak models. The corresponding static equation has been analytically solved by Kaneko and Takimoto (1982).

This paper is organized as follows: in Section 2 we outline the basic equations to be used, both magnetohydrodynamic and thermodynamic, and the magnetic field representation. In Section 3, we exploit Ampère's law to introduce the generalized Shafranov operator into the pressure equilibrium equation and obtain the general Maschke–Perrin equation. Section 4 presents an application to prolate spheroidal coordinates. Our conclusions are followed by an Appendix, where we outline some useful formulas involving curvilinear coordinates.

2. BASIC EQUATIONS

Let us consider an ideal plasma of electrons and singly charged ions in stationary equilibrium, where all partial time derivatives vanish, but allowing a constant nonzero velocity. The corresponding MHD equations are (Stacey, 1981)

$$\nabla\cdot(\rho\mathbf{v}) = 0 \tag{1}$$

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \mathbf{j}\times\mathbf{B} \tag{2}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{3}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} \tag{4}$$

$$\nabla\times\mathbf{E} = 0 \tag{5}$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0 \tag{6}$$

where

$$\rho = n(m_e + m_i) \tag{7}$$

is the mass density, n is the particle number density, and m_e, m_i are the electronic and ionic masses, respectively. $\mathbf{v}$, p, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{j}$ are the velocity, pressure, electric field, magnetic field, and plasma current density, respectively.

In order to get a closed system of field equations we must introduce some constitutive thermodynamic assumptions, as well as a working hypothesis on the plasma temperature. We suppose the plasma is an ideal gas, obeying the equation

$$p = \rho\bar{R}T = nkT \tag{8}$$

where T is the plasma temperature (sum of electronic and ionic temperatures), $\bar{R}$ and k being the gas constant and Boltzmann constant, respectively. We assume that the internal energy is simply proportional to the temperature.

Finally, we may suppose that the plasma electric conductivity is larger along the magnetic field lines than across them, in such a way that the plasma temperature turns out to be a surface quantity (Maschke and Perrin, 1980)

$$\mathbf{B}\cdot\nabla T = 0 \tag{9}$$

Let (x^1, x^2, x^3) denote the contravariant coordinates in a curvilinear coordinate system. In what follows, we will assume that $0 \leq x^3 \leq L$ is an ignorable coordinate with period L, i.e., surface quantities do not depend on x^3. Here $\hat{e}_i$ $(i = 1, 2, 3)$ are the covariant basis vectors for this system, such that $g_{ij} = \hat{e}_i\cdot\hat{e}_j$ are the covariant components of the metric tensor (see Appendix). We will consider only orthogonal coordinate systems, for which this tensor is diagonal, i.e., $g_{ij} = 0$ for $i \neq j$.

A magnetic field representation satisfying equation (3) can be written in terms of two scalar surface functions Ψ and I (transversal flux and current functions, respectively)

$$\mathbf{B}(x^1, x^2) = \frac{\hat{e_3}}{g_{33}} \times \nabla\Psi(x^1, x^2) - \mu_0 I(x^1, x^2)\frac{\hat{e_3}}{g_{33}} \tag{10}$$

We assume that the plasma is rotating with a constant angular rotation frequency Ω along the $\hat{e_3}$ direction and around the symmetry axis,

$$\mathbf{v}(x^1, x^2) = \Omega(x^1, x^2)\hat{e_3} \tag{11}$$

in such a way that mass conservation, equation (1), is identically satisfied.

An important consequence of the above formula appears when one combines the generalized Ohm's law, equation (6), with Faraday's law, equation (5). Using the representations for magnetic field and velocity given by (10) and (11), it follows that

$$\nabla\Omega \times \nabla\Psi = 0 \tag{12}$$

which is known as Ferraro's isorotation law. It implies that $\Omega = \Omega(\Psi)$, i.e., the angular frequency is a surface quantity in the sense that each flux surface rotates rigidly with a different frequency.

3. MASCHKE–PERRIN EQUATION IN CURVILINEAR COORDINATES

Substituting the magnetic field representation (10) in Ampére's law, equation (4), we have

$$\mathbf{j} = \frac{1}{\mu_0}\left[(\nabla\Psi\cdot\nabla)\left(\frac{\hat{e_3}}{g_{33}}\right) + \nabla^2\Psi\left(\frac{\hat{e_3}}{g_{33}}\right) + \mu_0\left(\frac{\hat{e_3}}{g_{33}}\right) \times \nabla I\right] \tag{13}$$

where we have used the fact that Ψ, I, and the metric tensor components do not depend on the ignorable coordinate x^3.

Let us introduce the so-called generalized Shafranov operator, defined for orthogonal coordinate systems as

$$\Delta^*\Psi = g_{33}\nabla\cdot\left(\frac{\nabla\Psi}{g_{33}}\right) \tag{14}$$

and substitute it into (13) to obtain

$$\mathbf{j} = \frac{1}{\mu_0}\left[\Delta^*\Psi\left(\frac{\hat{e_3}}{g_{33}}\right) + \mu_0\left(\frac{\hat{e_3}}{g_{33}}\right) \times \nabla I\right] \tag{15}$$

The Lorentz force term in the pressure equilibrium equation, equation (2), is obtained by combining (10) with (15), and reads

$$\mathbf{j} \times \mathbf{B} = \frac{1}{g_{33}}\left(\nabla I \times \nabla\Psi - \frac{1}{\mu_0}\Delta^*\Psi\nabla\Psi - \mu_0 I\nabla I\right) \tag{16}$$

whereas the convective part of the velocity time derivative is

$$\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \rho\Omega^2 \frac{\partial \hat{e}_3}{\partial x^3} \tag{17}$$

As $\partial\hat{e}_3/\partial x^3$ is orthogonal to $\hat{e}_3$, we can write $\partial\hat{e}_3/\partial x^3$ as a linear combination of the *contravariant* basis vectors $\hat{e}^1$ and $\hat{e}^2$. The coefficients of this linear combination are proportional to derivatives of g_{33}, such that $\partial\hat{e}_3/\partial x^3 = -(\nabla g_{33})/2$.

Substituting (17) and (16) in the pressure equilibrium equation (2) we observe that $\nabla I \times \nabla\Psi$ is null, since it would have only the $\hat{e}_3$ component, and there is no such component in the other terms of (2). The consequence is that $I = I(\Psi)$ is also a surface quantity, as is known in the static equilibrium case (Stacey, 1981). This implies that (2) is equivalent to the following equation:

$$\left(\Delta^*\Psi + \frac{\mu_0^2}{2}\frac{dI^2}{d\Psi}\right)\nabla\Psi = -\mu_0 g_{33}\left(\nabla p - \frac{\rho\Omega^2}{2}\nabla g_{33}\right) \tag{18}$$

Taking the gradient of equation (8) and summing and subtracting the term $kT\ln(\rho/\rho_0)$, where ρ_0 is a characteristic value of the plasma density, we can write the factor in the right-hand side of (18) in the following form:

$$\begin{aligned}\nabla p - \frac{\rho\Omega^2}{2}\nabla g_{33} &= n\nabla\left(kT\ln\frac{\rho}{\rho_0}\right) + nk\left(1 - \ln\frac{\rho}{\rho_0}\right)\frac{dT}{d\Psi}\nabla\Psi \\ &\quad -(m_e + m_i)n\left[\nabla\left(\frac{\Omega^2 g_{33}}{2}\right) - g_{33}\Omega\frac{d\Omega}{d\Psi}\nabla\Psi\right]\end{aligned} \tag{19}$$

where we have used equation (7) and the fact that both T and Ω are surface quantities.

The pressure equilibrium equation (2) is thus written in the form

$$\begin{aligned}&\left\{\Delta^*\Psi + \frac{\mu_0^2}{2}\frac{dI^2}{d\Psi} + \mu_0 g_{33} n\left[k\left(1 - \ln\frac{\rho}{\rho_0}\right)\frac{dT}{d\Psi} + (m_e + m_i)g_{33}\Omega\frac{d\Omega}{d\Psi}\right]\right\}\nabla\Psi \\ &= -\mu_0 g_{33} n\nabla\left(kT\ln\frac{\rho}{\rho_0} - \frac{(m_e + m_i)\Omega^2 g_{33}}{2}\right)\end{aligned} \tag{20}$$

and taking its vector product with $\nabla\Psi$, we get to the expression $\nabla\Theta \times \nabla\Psi = 0$, where we have defined

$$\Theta \equiv kT \ln \frac{\rho}{\rho_0} - \frac{(m_e + m_i)\Omega^2 g_{33}}{2} \tag{21}$$

which turns to be another surface quantity, i.e., $\Theta = \Theta\,(\Psi)$.

This enables us to define a function

$$G \equiv \frac{\rho_0 kT}{m_e + m_i}\, e^{\Theta/kT} = p \exp\left[\frac{-(m_e + m_i)\Omega^2 g_{33}}{2kT}\right] \tag{22}$$

which plays the role of a centrifugally corrected pressure p, since $G \to p$ as $\Omega \to 0$. Being also a surface quantity, we may take its derivative with respect to Ψ and isolate $d\Theta/d\Psi$. Substituting the result in (20), we have

$$\begin{aligned} &\left\{\Delta^*\Psi + \frac{\mu_0^2 dI^2}{2\, d\Psi} + \mu_0 g_{33} n \left[k\left(1 - \ln\frac{\rho}{\rho_0}\right)\frac{dT}{d\Psi} + (m_e + m_i) g_{33}\right.\right. \\ &\quad \left.\left. \times \left(kT\frac{d}{d\Psi}\left(\frac{\Omega^2}{2kT}\right) + \frac{\Omega^2}{2T}\frac{dT}{d\Psi}\right)\right]\right\}\nabla\Psi \\ &\quad = -\mu_0 g_{33} nk \left[\frac{T}{G}\frac{dG}{d\Psi} - \left(1 - \frac{\Theta}{kT}\right)\frac{dT}{d\Psi}\right]\nabla\Psi \end{aligned} \tag{23}$$

For an arbitrary nonvanishing $\nabla\Psi$ we obtain the form of Maschke–Perrin equation

$$\begin{aligned} \Delta^*\Psi = -\frac{\mu_0^2 dI^2}{2\, d\Psi} - \mu_0 g_{33} \exp\left[\frac{(m_e + m_i)\Omega^2 g_{33}}{2kT}\right] \\ \times \left[\frac{dG}{d\Psi} + G g_{33}(m_e + m_i)\frac{d}{d\Psi}\left(\frac{\Omega^2}{2kT}\right)\right] \end{aligned} \tag{24}$$

and in the limit of vanishing rotation we recover the Grad–Shafranov equation for static equilibria in a curvilinear coordinate system,

$$\Delta^*\Psi = -\frac{\mu_0^2}{2}\frac{dI^2}{d\Psi} - \mu_0 g_{33}\frac{dp}{d\Psi} \tag{25}$$

Solving the rotating plasma equation, as in the static case, requires prior knowledge of the Ψ dependence of G and I^2, which must be given as profiles.

Let us close this section by introducing the Mach number for the plasma rotation

$$\mathcal{M} = \frac{|v_{\langle 3 \rangle}|}{c_s} = \frac{\sqrt{g_{33}}\Omega}{\sqrt{\gamma RT}} \tag{26}$$

where $v_{\langle 3 \rangle}$ is the "physical" component of the velocity in the azimuthal direction (see Appendix), and c_s is the adiabatic sound velocity.

4. APPLICATION

In cylindrical coordinates $(x^1, x^2, x^3) = (R, Z, \varphi)$, equation (24) is essentially the same as that derived originally by Maschke and Perrin (1980). They also obtained a closed analytical solution for it by assuming that G and I^2 were linear functions of Ψ, and $d(\Omega^2/2kT)/d\Psi = 0$. Missiato and Sudano (1982) obtained an analytical solution in the form of an infinite series by assuming the same hypothesis for $\Omega^2/2kT$ and G, and considering I^2 quadratic in Ψ. Clemente and Farengo (1984) supposed that both I^2 and G were quadratic in Ψ, obtaining analytical and seminumerical (the angular part is factorized) solutions.

The corresponding equation in spherical coordinates $(x^1, x^2, x^3) = (r, \theta, \varphi)$ was derived by Viana *et al.* (1997), who also presented an analytical solution for it, assuming that G and I^2 are linear and quadratic in Ψ, respectively, as well as the same hypothesis for $\Omega^2/2kT$. The most important observed effect was the outward shift of the magnetic axis radial position with toroidal (azimuthal) plasma rotation.

One of the more convenient coordinate systems to study MHD equilibria in Spheromak-type and compact tori configurations is the prolate spheroidal system (Morse and Feshbach, 1953). In this case, we take the z axis as the symmetry axis of the plasma and use the contravariant coordinates $(x^1, x^2, x^3) = (\xi, \eta, \varphi)$, with

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = c\cosh\xi\cos\eta \tag{27}$$

where $0 \le \xi < \infty$, $0 \le \eta \le \pi$, $0 \le \varphi < L = 2\pi$, and $r = c\sinh\xi\sin\eta$, with $2c > 0$ being the distance between the two foci. The plasma surface is a coordinate surface $\xi = \xi_0$, which is a prolate spheroid of semimajor axis $c\cosh\xi_0$ and semiminor axis $c\sinh\xi_0$. The metric tensor components and related information are listed in the Appendix.

The magnetic field representation in such a case is [cf. equation (10)]

$$\mathbf{B}(\xi, \eta) = \frac{\hat{\varphi} \times \nabla\Psi(\xi, \eta) - \mu_0 I(\xi, \eta)\hat{\varphi}}{r(\xi, \eta)} \tag{28}$$

where $\hat{\varphi} \equiv \hat{e}_{\langle 3 \rangle}$ is the orthonormal basis vector in the azimuthal direction. The plasma rotation is described by the velocity profile $\mathbf{v}(\xi,\eta) = r(\xi, \eta)\Omega(\xi, \eta)\hat{\varphi}$, which satisfies Ferraro's isorotation law.

The generalized Shafranov operator in this coordinate system is

$$\Delta^*\Psi = \frac{1}{c^2(\cosh^2 \xi - \cos^2 \eta)}\left[\frac{\partial^2\Psi}{\partial\xi^2} + \frac{\partial^2\Psi}{\partial\eta^2} - \coth \xi \frac{\partial\Psi}{\partial\xi} - \cot \eta \frac{\partial\Psi}{\partial\eta}\right] \tag{29}$$

entering the Maschke–Perrin equation, which reads

$$\Delta^*\Psi = -\frac{\mu_0^2}{2}\frac{dI^2}{d\Psi} - \mu_0 c^2 \sinh^2 \xi \sin^2 \eta \exp\left[\frac{(m_e + m_i)\Omega^2 c^2 \sinh^2 \xi \sin^2 \eta}{2kT}\right]$$
$$\times\left[\frac{dG}{d\Psi} + Gc^2 \sinh^2 \xi \sin^2 \eta(m_e + m_i)\frac{d}{d\Psi}\left(\frac{\Omega^2}{2kT}\right)\right] \tag{30}$$

where the modified pressure function is given by

$$G = p \exp\left[\frac{-(m_e + m_i)\Omega^2 c^2 \sinh^2 \xi \sin^2 \eta}{2kT}\right] \tag{31}$$

The static case of this equation was studied by Kaneko and Takimoto (1982), who used profiles for p and I^2 linear and quadratic in Ψ, respectively. In our language and notation, they would correspond to the following profiles (a and h being positive constants):

$$G(\Psi) = \frac{a}{\mu_0}\Psi \tag{32}$$

$$\frac{dI^2(\Psi)}{d\Psi} = \frac{2h^2}{\mu_0^2 c^2}\Psi \tag{33}$$

$$\frac{\Omega^2}{kT} = \text{const} \tag{34}$$

which reduces the Maschke–Perrin equation to the particular form

$$\frac{\partial^2\Psi}{\partial\xi^2} + \frac{\partial^2\Psi}{\partial\eta^2} - \coth \xi \frac{\partial\Psi}{\partial\xi} - \cot \eta \frac{\partial\Psi}{\partial\eta} + h^2 (\cosh^2 \xi - \cos^2 \eta)\Psi$$
$$= -ac^4 \sinh^2 \xi \sin^2 \eta(\cosh^2 \xi - \cos^2 \eta) \exp\left(\frac{3}{2}\varepsilon \sinh^2 \xi \sin^2 \eta\right) \tag{35}$$

where

$$\varepsilon \equiv \frac{(m_e + m_i)\Omega^2 c^2}{3kT} = \frac{c^2}{g_{33}} \frac{\gamma}{3} \mathcal{M}^2 \tag{36}$$

is a measure of the rotational kinetic energy with respect to plasma thermal energy, and it is proportional to the square of Mach number (26).

An analytical solution to the static case ($\Omega = \varepsilon = 0$) was obtained by Kaneko and Takimoto (1982) as a combination of angular and radial spheroidal wave functions. The vacuum case of the static equation ($a = h = 0$) has a similar solution, written as a combination of Legendre and radial spheroidal wave functions.

5. CONCLUSIONS

In this paper we have derived an equation for MHD stationary equilibrium with azimuthal rotation in a curvilinear orthogonal coordinate system, according to a procedure introduced by Maschke and Perrin. The plasma configuration must have one ignorable coordinate, such that no surface quantities or metric coefficients may depend on it. The magnetic field is written in terms of two surface quantities that obey an elliptic partial differential equation, which in the limit of vanishing plasma rotation reduces to the Grad–Shafranov equation of static MHD equilibria. We have considered the plasma temperature as a surface quantity. The azimuthal rotation is such that every magnetic flux surfaces gyrates with a different angular frequency.

The equation so obtained generalizes previously known cases in cylindrical and spherical geometries. An application is given for prolate spheroidal coordinates, which is a system suitable for studies of compact tori configurations like the Spheromak. Solving this partial differential equation requires prior knowledge of two profiles: one for the current function and another for the centrifugally corrected pressure.

For cylindrical and spherical cases there are few analytical solutions available in the literature. For the proposed case of prolate spheroidal coordinates there is an analytical solution only for the static case. The rotating case turns out to be extremely difficult to handle analytically even by expanding the exponential term in powers of the dimensionless rotation parameter ε. This case would require the use of standard numerical methods similar to those used for solving the static Grad–Shafranov equation.

APPENDIX

In this appendix we will review some useful relations involving curvilinear and spheroidal coordinates. A more comprehensive treatment is found, for example, in D'haeseleer *et al.* (1991). Consider a system of contravariant coordinates (x^1, x^2, x^3). A coordinate surface is characterized by $x^i = const$, and a contravariant basis vector $\hat{e}^i = \nabla x^i$ is orthogonal to that surface.

A covariant basis vector is defined by $\hat{e}_i = \partial \mathbf{r}/\partial x^i$, where $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ ($\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are Cartesian basis vectors). The covariant and contravariant basis vectors are orthogonal in the sense that $\hat{e}_i \cdot \hat{e}^j = \delta_i^j$, where δ_i^j is the Kronecker delta.

The covariant metric tensor has components defined by $g_{ij} = \hat{e}_i \cdot \hat{e}_j$. An orthogonal coordinate system has a diagonal metric tensor, which means that

$$g = \det g_{ij} = g_{11} g_{22} g_{33}$$

The contravariant metric tensor also has only diagonal components for orthogonal systems, $g^{ii} = 1/g_{ii}$, such that $\det g^{ij} = 1/g$.

The relationship between covariant and contravariant basis vectors is also apparent in the following vector products (i, j, and k are in cyclic permutation of the indexes 1, 2, 3)

$$\hat{e}^i \times \hat{e}^j = \frac{1}{\sqrt{g}} \hat{e}_k \qquad \hat{e}_i \times \hat{e}_j = \sqrt{g}\, \hat{e}^k$$

The covariant and contravariant components of an arbitrary vector $\mathbf{A}$, given by $A_i = g_{ij} A^j$ and $A^i = g^{ij} A_j$, respectively (the summation convention is used), may or may not have the same physical dimensions of $\mathbf{A}$ itself, so it is convenient to define "physical" components of $\mathbf{A}$ and the corresponding orthonormal basis vectors as $\mathbf{A} = A_{\langle i \rangle} \hat{e}_{\langle i \rangle}$, where (no sum in $\bar{\imath}$)

$$A_{\langle i \rangle} = \frac{1}{\sqrt{g^{\bar{i}\bar{i}}}} A^{\bar{i}} = \sqrt{g_{\bar{i}\bar{i}}} A^{\bar{i}}, \qquad \hat{e}_{\langle i \rangle} = \frac{1}{\sqrt{g_{\bar{i}\bar{i}}}} \hat{e}_{\bar{i}} = \sqrt{g^{\bar{i}\bar{i}}} \hat{e}_{\bar{i}}$$

Let ϕ and $\mathbf{A}$ denote arbitrary smooth scalar and vector functions of x^i, respectively. Their gradient and divergence are defined as

$$\nabla\phi = \frac{\partial \phi}{\partial x^i} \hat{e}^i, \qquad \nabla \cdot \mathbf{A} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} A^i)$$

respectively; and the rotational is particularly simple if we require that x^3 is an ignorable coordinate, and reads

$$\nabla \times \mathbf{A} = \frac{1}{\sqrt{g}} \left[\frac{\partial A_3}{\partial x^2} \hat{e}_1 - \frac{\partial A_3}{\partial x^1} \hat{e}_2 + \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \right) \hat{e}_3 \right]$$

The covariant basis vectors of the prolate spheroidal coordinate system are given by

$$\hat{e}_\xi = c \sin \eta \cosh \xi \,(\cos \varphi \hat{\imath} + \sin \varphi \hat{\jmath}) + c \cos \eta \sinh \xi \hat{k}$$

$$\hat{e}_\eta = c \cos \eta \sinh \xi \,(\cos \varphi \hat{\imath} + \sin \varphi \hat{\jmath}) - c \sin \eta \cosh \xi \hat{k}$$

$$\hat{e}_\varphi = -c \sin \eta \sinh \xi \,(\sin \varphi \hat{\imath} - \cos \varphi \hat{\jmath})$$

in such a way that the nonzero covariant metric tensor components are

$$g_{11} = g_{22} = c^2 (\cosh^2 \xi - \cos^2 \eta), \qquad g_{33} = r^2 = c^2 \sinh^2 \xi \sin^2 \eta$$

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